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Fuzziness in Abacus Logic

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The concept of "abacus logic" has recently been developed by the author (Malhas, n.d.). In this paper the relation of abacus logic to the concept of fuzziness is explored. It is shown that if a certain "regularity" condition is met, concepts from fuzzy set theory arise naturally within abacus logics. In particular it is shown that every abacus logic then has a "pre-Zadeh orthocomplementation". It is also shown that it is then possible to associate a fuzzy set with every proposition of abacus logic and that the collection of all such sets satisfies natural conditions expected in systems of fuzzy logic. Finally, the relevance to quantum mechanics is discussed.

1. WHAT IS AN ABACUS?

The aim of this paper is to present a rich source of examples of logics akin to quantum logic and to fuzzy logic. In a textbook on quantum logic (or quantum fuzzy logic), the ideas in this paper would find their rightful place not in the main text, but in the examples and exercises. This is not at all bad. The examples and exercises in a textbook help to motivate the discussion and to show that the ideas in the main text are natural, useful, and that they have applications beyond the original circumstance in which they were first conceived.

The objects under study in this paper are functions

$$\varphi: \quad \Gamma \to \mathscr{P}(\Re)\setminus\{\varnothing\}$$

where Γ is a nonempty set and $\mathscr{P}(\Re)$ is the set of all subsets of the real line $\Re$. If Γ is finite, say $\Gamma = \{a, b, c, d, e\}$, then we can picture φ as follows: First we draw five parallel copies of $\Re$, one for each element of Γ. Next, for each $i \in \Gamma$, we picture $\varphi(i)$ as a nonempty set of points on the ith copy of $\Re$. It is natural, in view of such a picture, to think of $\varphi(i)$ as the set of

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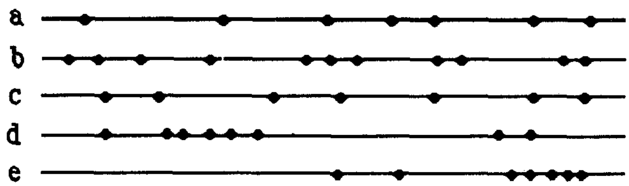

Fig. 1.

"beads" on the ith "rung" of an "abacus" (Figure 1). In fact we shall find this intuitive picture useful for our purposes.

Definition. Any function

$$\varphi: \quad \Gamma \to \mathscr{P}(\mathfrak{R}) \backslash \{\varnothing\}$$

is here called an *abacus on* Γ. The elements of Γ are called the *rungs* of φ and, for every $i \in \Gamma$, $\varphi(i)$ is called *the set of beads* on the ith rung of φ.

We shall show that with every "regular" collection of abaci on a nonempty set Γ we can associate a fuzzy logic called "fuzzy abacus logic" and that quantum logic is a special case of such a logic. The idea of "abacus logic" was first studied (Malhas, n.d.) without reference to pseudonegation or fuzziness. The next section contains a quick review, with new examples, of the idea of an "abacus logic." The last two sections deal with fuzziness and application to quantum theory.

2. ABACUS LOGIC

Let φ be an abacus on Γ and let $\mathbf{B}$ be the collection of Borel subsets of $\mathfrak{R}$. For any $E \in \mathbf{B}$, it is either true or false that the beads on the ith rung are all within E. That is, it is either true or false that $\varphi(i) \subseteq E$.

Let $\mathbf{K}$ be a nonempty collection of abaci on Γ. With every $\varphi \in \mathbf{K}$ let $\tau_\varphi: \Gamma \times \mathbf{B} \to \{0, 1\}$ be given by $\tau_\varphi(i, E) = 1$ iff $\varphi(i) \subseteq E$. We can therefore think of the pair (i, E) as a symbol representing the English statement

The beads on the ith rung are all within the Borel set E.

This statement does not refer to a particular abacus in $\mathbf{K}$. It is true for the abacus φ iff $\varphi(i) \subseteq E$, i.e., iff $\tau_\varphi(i, E) = 1$, otherwise $\tau_\varphi(i, E) = 0$. Note that $\tau_\varphi(i, E) = 0$ *does not mean that* $\tau_\varphi(i, E') = 1$ *or, equivalently, that* $\varphi(i) \subseteq E'$, *but merely that* $\varphi(i) \nsubseteq E$. The elements of $\Gamma \times \mathbf{B}$ shall here be called *elementary formulas*. If $\tau_\varphi(i, E) = 1$, then we shall say that the elementary formula (i, E) *is true on* the abacus φ. For every elementary formula $u = (i, E)$ we let $\mathbf{K}(u)$ be the set of all abaci in $\mathbf{K}$ on which u is true.

The collection $\mathbf{K}$ of abaci determines an equivalence relation $\approx_{\mathbf{K}}$ on $\Gamma \times \mathbf{B}$ defined by

$$(i, E) \approx_{\mathbf{K}} (j, G) \quad \text{iff} \quad \forall \varphi \in \mathbf{K}, \quad \tau_\varphi(i, E) = \tau_\varphi(j, G)$$

The equivalence class of the elementary formula (i, E) shall be called a *proposition* and be denoted by $[(i, E)]$. Let $\mathscr{V}$ be the set of all propositions. Then $\mathscr{V}$ is partially ordered by $[(i, E)] \leq_{\mathbf{K}} [(j, G)]$ iff $\forall \varphi \in \mathbf{K}$, $\tau_\varphi(i, E) \leq \tau_\varphi(j, G)$. The partially ordered set $(\mathscr{V}, \leq_{\mathbf{K}})$ shall be called *the logic of* $\mathbf{K}$ or an *abacus logic*. Every abacus logic has a first element $\mathbf{o}$ and a last element $\mathbf{i}$ defined as follows: For any rung i, put

$$\mathbf{o} = [(i, \varnothing)] \qquad \text{and} \qquad \mathbf{i} = [(i, \mathfrak{R})]$$

It follows that $\mathbf{o}, \mathbf{i}$ are well defined, $\mathbf{o} \neq \mathbf{i}$, and that for every $\alpha \in \mathscr{V}$, $\mathbf{o} \leq_{\mathbf{K}} \alpha \leq_{\mathbf{K}} \mathbf{i}$. It is clear that $(i, E) \approx_{\mathbf{K}} (j, G)$ iff $\mathbf{K}(i, E) = \mathbf{K}(j, G)$. In fact, we have the following theorem.

Theorem 2.1. For all elementary formulas $u, v \in \Gamma \times \mathbf{B}$:
(i) $[u] \leq_{\mathbf{K}} [v]$ iff $\mathbf{K}(u) \subseteq \mathbf{K}(v)$.
(ii) If $[u] \neq [v]$, then $\mathbf{K}(u) \neq \mathbf{K}(v)$.
(iii) If $u = (i, E)$, then $\mathbf{K}(i, E') \subseteq (\mathbf{K}(i, E))'$.

Proof. (i) and (ii) are obvious. For (iii) suppose that $\varphi \in \mathbf{K}(i, E')$. Then $\tau_\varphi(i, E') = 1$. This implies $\varphi(i) \subseteq E'$, which implies $\varphi(i) \nsubseteq E$, which implies $\tau_\varphi(i, E) = 0$, which implies that $\varphi \notin \mathbf{K}(i, E)$, which implies $\varphi \in (\mathbf{K}(i, E))'$ ■

This theorem is important not only for what it says but also *for what it does not say*. Parts (i) and (ii) say that there is an imbedding of abacus logic, a partially ordered set, into the partially ordered set $(\mathscr{P}(\mathbf{K}), \subseteq)$ of subsets of $\mathbf{K}$. Of course $(\mathscr{P}(\mathbf{K}), \subseteq)$ is also a complemented distributive lattice. The theorem does *not* say that abacus logic is isomorphic to a sublattice of $(\mathscr{P}(\mathbf{K}), \subseteq)$. the example below shows that even if abacus logic is a lattice, it need not be isomorphic to a sublattice of $\mathscr{P}(\mathbf{K})$. This is neither surprising nor profound: Two elements in a subset $\mathscr{C}$ of $\mathscr{P}(\mathbf{K})$ need not have either a g.l.b. or an l.u.b. in $\mathscr{C}$ even though their g.l.b. and l.u.b. exist in $\mathscr{P}(\mathbf{K})$. Furthermore, if two elements have an l.u.b. in $\mathscr{C}$, then this need not be their l.u.b. in $\mathscr{P}(\mathbf{K})$. In fact, their l.u.b. in $\mathscr{C}$ will, generally speaking, be "above" (i.e., contains) their l.u.b. in $\mathscr{P}(\mathbf{K})$. Similarly, if their g.l.b. exists in $\mathscr{C}$, it is "lower than" (i.e., a subset of) their g.l.b. in $\mathscr{P}(\mathbf{K})$. Part (iii) of the theorem tells us that also $\mathbf{K}(i, E')$ cannot be identified with the complement in $\mathbf{K}$ of $\mathbf{K}(i, E)$.

Theorem 2.1 also has practical importance when **K** is finite. For example, suppose **K** has exactly three elements. Then $(\mathscr{P}(\mathbf{K}), \subseteq)$ has the famous Hasse diagram of Figure 2. Then regardless of how many rungs there are or how the beads are distributed on the various rungs, the abacus logic associated with **K** is isomorphic to a subset of this diagram. To construct the Hasse diagram of abacus logic we use part (ii) of Theorem 2.1. For every elementary formula (i, E) we search Figure 2 for the element which corresponds to $\mathbf{K}(i, E)$. The set of all elements that we find in this way is the desired abacus logic.

Example. Now let us show that abacus logic can be a lattice and that then it need not be distributive. We shall also illustrate the combinatorial arguments that are needed in connection with a finite collection of abaci with finite sets of beads on the rungs. these arguments are spelled out in tedious detail because the topic is new and one has to learn how to handle it. Let **K** be the collection of abaci, on $\Gamma = \{a, b\}$, in Figure 3.

The positions of the beads on rung "a" are chosen from the set $A = \{-2, 1, 2\}$. Thus for every Borel set E, $\mathbf{K}(\mathrm{a}, E) = \mathbf{K}(\mathrm{a}, E \cap A)$. We also note that $\mathbf{K}(\mathrm{a}, \{-2, 2\}) = \mathbf{K}(\mathrm{a}, \{-2\}) = \mathbf{K}(\mathrm{a}, \{-2, 1\}) = \{\varphi_2\}$. Also $\mathbf{K}(\mathrm{a}, \{1\}) = \mathbf{K}(\mathrm{a}, \{2\}) = \mathbf{K}(\mathrm{a}, \varnothing) = \varnothing$, since there does not exist an abacus φ in the collection for which $\varphi(\mathrm{a}) \subseteq \{1\}$ or $\varphi(\mathrm{a}) \subseteq \{2\}$. Thus every pair (a, E) is equivalent, under $\approx_{\mathbf{K}}$ to exactly one of the following four pairs:

$$(\mathrm{a}, \varnothing), \quad (\mathrm{a}, \{-2, 1, 2\}), \quad (\mathrm{a}, \{-2\}), \quad (\mathrm{a}, \{1, 2\})$$

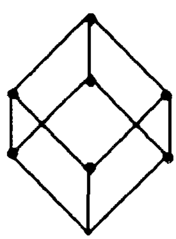

Fig. 2.

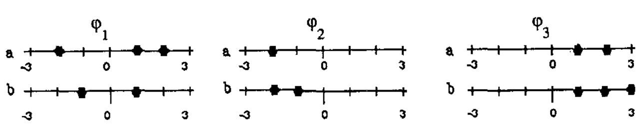

Fig. 3.

Similarly, for every Borel set E, every pair (b, E) is equivalent, under $\approx_{\mathbf{K}}$ to exactly one of the following seven pairs:

$$(\mathrm{b}, \varnothing), \quad (\mathrm{b}, \{-1, 1\}), \quad (\mathrm{b}, \{-2, -1\}), \quad (\mathrm{b}, \{1, 2, 3\})$$
$$(\mathrm{b}, \{-2, -1, 1\}), \quad (\mathrm{b}, \{-1, 1, 2, 3\}), \quad (\mathrm{b}, \{-2, -1, 1, 2, 3\})$$

We also have the following "mixed" equivalences:

$$(\mathrm{a}, \{1, 2\}) \approx_{\mathbf{K}} (\mathrm{b}, \{1, 2, 3\}), \qquad (\mathrm{a}, \{-2\}) \approx_{\mathbf{K}} (\mathrm{b}, \{-2, -1\}),$$
$$(\mathrm{b}, \varnothing) \approx_{\mathbf{K}} (\mathrm{a}, \varnothing)$$
$$(\mathrm{a}, \mathfrak{R}) \approx_{\mathbf{K}} (\mathrm{a}, \{-2, 1, 2\}) \approx_{\mathbf{K}} (\mathrm{b}, \{-2, -1, 1, 2, 3\}) \approx_{\mathbf{K}} (\mathrm{b}, \mathfrak{R})$$

It is sufficient to show how the first equivalence is obtained: For every abacus φ in the collection, $\varphi(\mathrm{a}) \subseteq \{1, 2\}$ iff $\varphi = \varphi_3$ iff $\varphi(\mathrm{b}) \subseteq \{1, 2, 3\}$. Thus the set of all propositions is a set of seven elements:

$$\mathbf{o} = [(\mathrm{b}, \varnothing)], \qquad \mathbf{i} = [(\mathrm{b}, \mathfrak{R})], \qquad [(\mathrm{b}, \{1, 2, 3\})], \qquad [(\mathrm{b}, \{-2, -1\})]$$
$$[(\mathrm{b}, \{-\mathrm{q}, 1\})], \qquad [(\mathrm{b}, \{-1, 1, 2, 3\})], \qquad [(\mathrm{b}, \{-2, -1, 1\})].$$

It is obvious that for a (fixed) rung i and any Borel sets E, G, we have $E \subseteq G$ implies $[(i, E)] \leq_{\mathbf{K}} [(i, G)]$. Thus the Hasse diagram of Figure 4 represents the corresponding abacus logic. This is a lattice, but not a distributive lattice: Try the distributive law with $(\alpha \vee \gamma) \wedge \delta$. ■

Define *pseudonegation* to be the operation $\#: \Gamma \times \mathbf{B} \to \Gamma \times \mathbf{B}$ given by $\#(i, E) = (i, E')$. This operation, which suggests a sort of "negation," does not necessarily "behave well" with $\approx_{\mathbf{K}}$. That is, if $(i, E) \approx_{\mathbf{K}} (j, G)$, then it does not follow that $(i, E') \approx_{\mathbf{K}} (j, G')$. In the last example, take

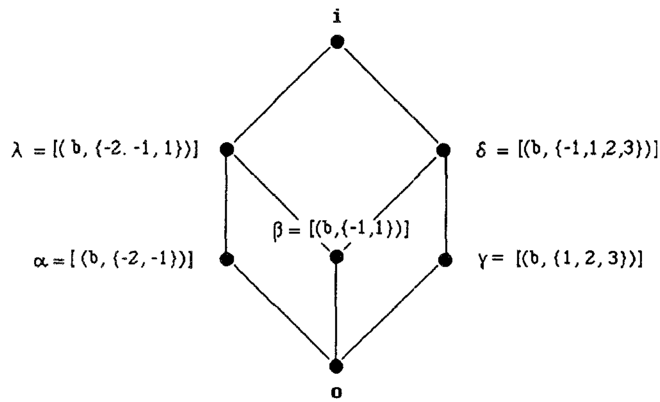

Fig. 4.

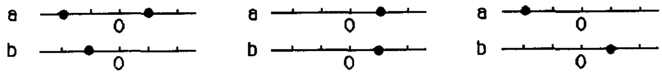

Fig. 5.

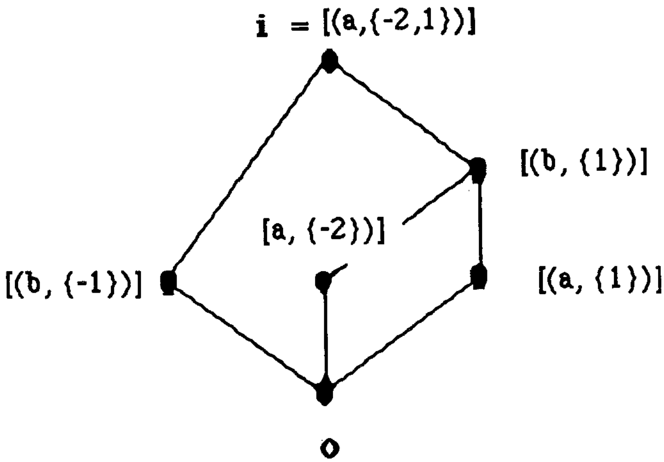

Fig. 6.

$E=\{-2,1\}$ and $G=\{-2,-1\}$. Then $(a,E) \approx_{\mathbf{K}} (b,G)$. But $[(a,E')]=[(a,\{-1\})]=\mathbf{o}$, whereas $[(b,G')]=(b,\{1,2,3\}) \neq \mathbf{o}$. A set $\mathbf{K}$ of abaci for which pseudo negation "behaves well" with $\approx_{\mathbf{K}}$ shall be called *regular*. The collection of Figure 5 is regular. In fact the equivalence classes of $(\mathrm{a},\{1\}), (\mathrm{a},\{-2\}), (\mathrm{b},\{1\}), (\mathrm{b},\{-1\})$ are distinct. Also we have $(\mathrm{b},\mathfrak{R}) \approx_{\mathbf{K}} (\mathrm{b},\{-1,1\}) \approx_{\mathbf{K}} (\mathrm{a},\{-2,1\}) \approx_{\mathbf{K}} (\mathrm{a},\mathfrak{R})$. From these observations the regularity of $\mathbf{K}$ easily follows. Abacus logic for this collection has the Hasse diagram of Figure 6.

3. FUZZY ABACUS LOGIC

As promised in the Introduction, we shall now use the concept of abacus to illustrate various aspects of fuzzy logic. From here on we shall be concerned only with regular collections of abaci.

(a) Zadeh posets. If $\mathbf{K}$ is regular, then the operation $*: \mathscr{V} \to \mathscr{V}$ given by $[(i,E)]^*=[(i,E')]$ is well defined. Before proving the next theorem it is useful to point out that the condition $\alpha \leq_{\mathbf{K}} \alpha^*$, where $\alpha=[(i,E)]$, is equivalent to the condition that for all $\varphi \in \mathbf{K}$, if $\varphi(i) \subseteq E$, then $\varphi(i) \subseteq E'$,

which implies that for all $\varphi \in \mathbf{K}$, $\varphi(i) \nsubseteq E$ or $\varphi(i) \subseteq E'$, which implies that for all $\varphi \in \mathbf{K}$, $\tau_\varphi(i, E) = 0$. The $\alpha = [(i, E)] = \mathbf{o}$.

Theorem 3.1. The operation * is a *pre-Zadeh orthocomplementation* of abacus logic, i.e., for all $\alpha, \beta \in \mathscr{V}$, (i) $\alpha^{**} = \alpha$ and (ii) if $\alpha \leq_{\mathbf{K}} \alpha^*$ and $\beta^* \leq_{\mathbf{K}} \beta$, then $\alpha \leq_{\mathbf{K}} \beta$.

Proof. Part (i) is obvious. Part (ii) is trivially true since, by the comments preceding the theorem, $\alpha = \mathbf{o}$. ■

To become a *Zadeh orthocomplementation,* * must also satisfy the condition that if $\alpha \leq_{\mathbf{K}} \beta$, then $\beta^* \leq_{\mathbf{K}} \alpha^*$ (Cattaneo and Nisticò, 1989). This, however, need not be satisfied by the abacus logic of a regular collection of abaci. Figure 5 depicts a regular collection of abaci. Let $\alpha = [(\mathrm{a}, \{1\})]$, $\beta = [(\mathrm{b}, \{1\})]$. Then $\alpha^* = [(\mathrm{a}, \{1\}')] = [(\mathrm{a}, \{-2\})]$ and, similarly, $\beta^* = [(\mathrm{b}, \{-1\})]$ and we have $\alpha \leq_{\mathbf{K}} \beta$, but $\beta^* \nleq_{\mathbf{K}} \alpha^*$, as can be verified from Figure 6. One can easily construct examples of an abacus logic which satisfies if $\alpha \leq_{\mathbf{K}} \beta$, then $\beta^* \leq_{\mathbf{K}} \alpha^*$. We shall meet one such example when we discuss quantum mechanics. The next conjecture is stated without further comment.

Conjecture. Every Zadeh poset in which $\alpha \leq_{\mathbf{K}} \alpha^*$ implies that $\alpha = \mathbf{o}$ is an abacus logic.

(b) Fuzziness. Each English statement, "*The beads on the ith rung are within E,*" is essentially a *fuzzy statement:* The beads on the ith rung of an abacus can all be within E or they can all be outside E or they can be partly inside E and partly outside E. The function τ_φ gives a very coarse estimate of the depth of penetration of $\varphi(i)$ into E: If $\varphi(i)$ is entirely within E, then $\tau_\varphi(i, E) = 1$. All other degrees of penetration of $\varphi(i)$ into E are given degree of penetration 0, i.e., if $\varphi(i) \nsubseteq E$, then $\tau_\varphi(i, E) = 0$. Suppose $\mathbf{K}$ is a regular collection of abaci with abacus logic $(\mathscr{V}, \leq_{\mathbf{K}})$ and let $\rho_\varphi : \mathscr{V} \to [0, 1]$ be such that:

1. $\rho_\varphi[(i, E)] = 1$ iff $\varphi(i) \subseteq E$ and $\rho_\varphi[(i, E)] = 0$ iff $\varphi(i) \subseteq E'$.
2. $\rho_\varphi[(i, E')] = 1 - \rho_\varphi[(i, E)]$.

Such a function exists, e.g., set $\rho_\varphi[(i, E)] = 1/2$ if neither $\varphi(i) \subseteq E$ nor $\varphi(i) \subseteq E'$. Then the function $\omega_\varphi : \Gamma \times \mathbf{B} \to [0, 1]$ given by $\omega_\varphi(i, E) = \rho_\varphi[(i, E)]$ is a finer measure of the penetration of $\varphi(i)$ into E. We may call ρ_φ *a fuzziness measure.*

For every $\alpha \in \mathscr{V}$ define $f_\alpha : \mathbf{K} \to [0, 1]$ by setting $f_\alpha(\varphi) = \rho_\varphi(\alpha)$. Then f_α is a fuzzy set on $\mathbf{K}$. It has the property that if $\alpha \neq \beta$, then $f_\alpha(\varphi) \neq f_\beta(\varphi)$. Clearly, $f_{\mathbf{o}} = 0$, the zero function. Let $\mathscr{F}$ be the collection of all fuzzy sets

g on $\mathbf{K}$ such that $g = f_\alpha$ for some α in $\mathscr{V}$. Then $\mathscr{F}$ has the following properties:

$\mathscr{F}1$. $f_0 \in \mathscr{F}$.
$\mathscr{F}2$. If $g \in \mathscr{F}$, then $1 - g \in \mathscr{F}$.
$\mathscr{F}3$. For all $g \in \mathscr{F}$, $g \leq 1 - g$ implies $g = 0$.

We may call $\mathscr{F}$ *fuzzy abacus logic*. If abacus logic is a lattice, then "sums" and "products" of elements of $\mathscr{F}$ are defined and belong to $\mathscr{F}$:

$\mathscr{F}4$. $f_\alpha \oplus f_\beta(\varphi) = f_{\alpha \vee \beta}(\varphi)$ and $f_\alpha \otimes f_\beta(\varphi) = f_{\alpha \wedge \beta}(\varphi)$.

The result would then be a variation on the concept of a *generalized quantum logic* as defined by Pykacz (1992).

4. RELEVANCE TO QUANTUM MECHANICS

With every *finite-dimensional* Hilbert space $\mathscr{H}$ over the complex numbers we can associate a collection of abaci such that abacus logic is isomorphic to the lattice of subspaces of $\mathscr{H}$. See Malhas (n.d.).

1. *The Rungs.* Let Γ be the set of all self-adjoint operators on $\mathscr{H}$. In quantum mechanics self-adjoint operators are called *observables. The set of all rungs is the set of all observables.*

2. *The Beads.* Let s be a one-dimensional subspace of $\mathscr{H}$. In quantum mechanics such subspaces are called *pure states*. Let i be an observable. For the purposes of this paper, a set of eigenvalues of i shall be called *good for* s if the subspace spanned by the corresponding eigenvectors contains s. Clearly, the set of *all* eigenvalues of i is good for s. The empty set is not good for s. The intersection of any collection of good sets is also a good set. Let $\psi_s(i)$ be *the intersection of all sets which are good for s*. Thus $\psi_s(i) \neq \varnothing$ and, hence, the function $\varphi: \Gamma \to \mathscr{P}(\Re) \backslash \varnothing$ defined by $\varphi(i) = \psi_s(i)$ is an abacus. The set of beads on the ith rung is $\psi_s(i)$. Let $\mathbf{K}$ be the set of all abaci obtained in this way. Thus $\varphi \in \mathbf{K}$ iff $\varphi(i) = \psi_s(i)$ for some state s. The next theorem tells us that we can identify pure states with abaci. Thus $\mathbf{K}$ contains many abaci.

Theorem 4.1. The correspondence $s \to \psi_s$ is $1-1$.

Proof. For every state s let i_s be the orthogonal projection onto s. Then i_s has a set of two eigenvalues $\{0, 1\}$. Let z be a state other than s. It easily follows from the definitions of ψ_s and ψ_z that $\psi_z(i_z) = \{1\}$, but $\psi_s(i_z) = \{0\}$ if $s \perp z$, or $\psi_s(i_z) = \{0,1\}$. Thus $\psi_s \neq \psi_z$. ■

With $\mathbf{K}$ there is associated an abacus logic. It remains to show that abacus logic, in this case, is regular and that it is isomorphic to the lattice of subspaces of $\mathscr{H}$. By definition (see Section 1), for every pure state s, $\tau_{\psi_s}(i, E) = 1$ iff $\psi_s(i) \subseteq E$.

For every observable i and $E \in \mathbf{B}$, let the symbol $i(E)$ denote the subspace spanned by the set of all eigenvectors of the eigenvalues of i that happened to be in E.

Theorem 4.2. For every $E \in \mathbf{B}$, $\tau_{\psi_s}(i, E) = 1$ iff $s \subseteq i(E)$.

Proof. First suppose $s \subseteq i(E)$. Then the set of eigenvalues of i that are in E is good for s. Thus $\psi_s(i) \subseteq E$ and therefore that $\tau_{\psi_s}(i, E) = 1$. Conversely, suppose that $\tau_{\psi_s}(i, E) = 1$. Then $\psi_s(i) \subseteq E$. But $\psi_s(i)$ is a set of eigenvalues good for s. Hence, $s \subseteq i(E)$. ■

Theorem 4.3. $[(i, E)] \leq_{\mathbf{K}} [(j, G)]$ iff $i(E) \subseteq j(G)$.

Proof. By definition, $[(i, E)] \leq_{\mathbf{K}} [(j, G)]$ iff, for every state s, $\tau_{\psi_s}(i, E) \leq \tau_{\psi_s}(j, G)$ iff $\tau_{\psi_s}(i, E) = 1$ implies $\tau_{\psi_s}(j, G) = 1$ iff, by the last theorem, for every state s, $s \subseteq i(E)$ implies $s \subseteq j(G)$ iff $i(E) \subseteq j(G)$. ■

Theorem 4.4. $\mathbf{K}$ is regular.

Proof. Using the last theorem, $[(i, E] = [(j, G))]$ iff $i(E) = j(G)$ iff $i(E') = j(G')$ iff $[(i, E')] = [(j, G')]$. ■

For every subspace A let p_A be the orthogonal projection on A.

Theorem 4.5. Abacus logic is a lattice isomorphic to the lattice of subspaces of $\mathscr{H}$.

Proof. See Malhas (n.d.).

Thus we may identify abacus logic with quantum logic. Each proposition $[(i, E)]$ may now be called a *quantum proposition.* Gleason's theorem characterizes probability measures on abacus logic. The probability of the quantum proposition $[(i, E)]$ in the state s is given by $\rho_s[(i, E)] = \text{trace}(p_s p_{i(E)})$. In abacus logic, probability induces a natural fuzziness measure. For every pure state s let $\omega_{\psi_s}: \Gamma \times \mathbf{B} \to [\mathbf{0}, \mathbf{1}]$ given by $\omega_{\psi_s}(i, E) = \text{trace}(p_s p_{i(E)})$. As in Section 3, the unique fuzzy set (on $\mathbf{K}$) corresponding to the quantum proposition $\alpha = [(i, E)]$ is $f_\alpha(\psi_s) = \rho_s(\alpha) = \text{trace}(p_s p_{i(E)})$.

NOTE ADDED IN PROOF

Let Q be the set of all functions $f_\alpha: \mathbf{K} \to [0, 1]$ given by $f_\alpha(\psi_S) = \text{trace}(p_S p_{i(E)})$. Then Q satisfies conditions $\mathscr{F}_1$ to $\mathscr{F}_4$ above because the underlying set $\mathbf{K}$ is regular and because abacus logic, in this case, is a lattice.

It should, however, be noted that the appropriate partial order relation in a system $\mathscr{F}$ of fuzzy sets satisfying $\mathscr{F}_1$ to $\mathscr{F}_4$ is not the "natural" one,

$f_\alpha \leq f_\beta$ iff $f_\alpha(\psi_S) \leq f_\beta(\psi_S)$ for every s, but the partial order relation inherited from abacus logic (which is now assumed to be a lattice)

$$f_\alpha \leq f_\beta \text{ iff } \alpha \leq_{\mathbf{K}} \beta$$

In this way we turn $\mathscr{F}$ into a lattice in which $\oplus$ and $\otimes$ are the lattice operations of join and meet, respectively. The partial order relation inherited from abacus logic is the one which is "compatible", in an obvious sense, with the lattice operations in abacus logic, and if we define $f_\alpha^\perp$ to be $1 - f_\alpha$, then it easily follows that $^\perp$ is something like an orthocomplementation, in fact a pre-Zadeh orthocomplementation.

In Q, the operation $^\perp$ is an orthocomplementation and Q is an orthocomplemented lattice isomorphic to the orthocomplemented lattice of subspaces of $\mathscr{H}$. It also turns out that in Q the "natural" partial ordering and the abacus logic ordering are the same.

REFERENCES

Cattaneo, G., and Nisticò, G. (1989). Brower–Zadeh posets and three-valued Lukasiewicz posets, *Fuzzy Sets and Systems*, **33**, 165–190.

Malhas, O. (n.d.). Abacus logic, *Journal of Symbolic Logic*, to appear.

Pykacz, J. (1992). Generalized fuzzy quantum logics, Reprint No. 84, Instytut Matematyki, Uniwersytet Gdanski, Poland.